

# Mixed identities and Neumann's Lemma

Paolo Marimon  
joint work with Michael Pinski

University of Oxford

LyMoTs meeting, University of Leeds, 15 June 2026



Funded by UKRI ERC Guarantee Grant New Approaches to Approximability of Satisfiable Problems.

# Outline

- ① Background
  - Equations in groups
  - Oligomorphicity
  - Mixed identities
- ② On algebraic closure
- ③ The result
- ④ Strategy
- ⑤ Bibliografy

## Words with constants

$G$  := a group.

### Definition (Word with constants)

A **word with constants** for  $G$  in  $r$  many variables is

$$w(x_1, \dots, x_r) := \gamma_n x_{\iota_n}^{\epsilon_n} \gamma_{n-1} x_{\iota_{n-1}}^{\epsilon_{n-1}} \dots \gamma_1 x_{\iota_1}^{\epsilon_1} \gamma_0$$

for  $\gamma_0, \dots, \gamma_n \in G$ ,  $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ , and  $\iota_1, \dots, \iota_n \in \{1, \dots, r\}$ .

I.e., a **term** in  $\mathcal{L} := \{\cdot, {}^{-1}, (\gamma)_{\gamma \in G}\}$ .

We ask that  $w$  is **reduced**:  $\iota_{j-1} = \iota_j$  and  $\epsilon_{j-1} = -\epsilon_j \Rightarrow \gamma_j \notin Z(G)$ .

$w$  is **singular** if “forgetting constants” and reducing we get 1:

$$y^{-1} x \gamma_2 x^{-1} \gamma_1 y \xrightarrow[\text{constants}]{\text{forget}} y^{-1} x x^{-1} y \mapsto y^{-1} y \mapsto 1.$$

💡 Singular words are “close” to the identity;

Otherwise,  $w$  is **regular** (a.k.a. non-singular).

# Equations in groups

$w$  defines a **word map**:  $w : G^r \rightarrow G (g_1, \dots, g_r) \mapsto w(g_1, \dots, g_r)$ .

We are interested in **equations**  $w(x_1, \dots, w_r) = 1$ .

- when can we **solve**  $w$ ?

$$\exists g_1, \dots, g_r \in G \text{ s.t. } w(g_1, \dots, g_r) = 1 ;$$

- when is  $w$  a **mixed identity**?

$$\forall g_1, \dots, g_r \in G w(g_1, \dots, g_r) = 1 .$$

-  These notions are in tension!

$$\forall \bar{g} \in G w(\bar{g}) = 1 \Rightarrow \text{for } \gamma \in G \setminus \{1\}, \nexists \bar{g} \in G \gamma w(\bar{g}) = 1.$$

# Solving equations

## Conjecture (Klyachko and Thom 2017)

$G$  a group,  $w(x_1, \dots, x_r)$  regular.

Then, there is  $H \supseteq G$  s.t.  $\exists h_1, \dots, h_r \in G$   $w(h_1, \dots, h_r) = 1$ .

- $r = 1$  this is the Kervaire-Laudenbach Conjecture:
  - Gerstenhaber and Rothaus 1962: true for all finite groups;
  - Pestov 2008: true for all sofic groups.

Interest in finding groups for which  $w(G^r)$  is large (possibly  $= G$ ).

(because  $\gamma \notin w(G^r) \Leftrightarrow 1 \notin \gamma^{-1}w(G^r)$ )

# Solving equations

## Conjecture (Klyachko and Thom 2017)

$G$  a group,  $w(x_1, \dots, x_r)$  **regular**.

Then, there is  $H \supseteq G$  s.t.  $\exists h_1, \dots, h_r \in G$   $w(h_1, \dots, h_r) = 1$ .

## Theorem (Adeleke and Holland 1994)

$G := \text{Aut}(\mathbb{Q}; <)$ .  $w$  **with no constants and non-trivial**.

Then,  $w(G^r) = G$ .

## Theorem (Mycielski 1987; Lyndon 1990)

$G := \text{Sym}(\mathbb{N})$ .  $w$  **with no constants and**  $w \neq x^n$  ( $n \geq 2$ ).

Then,  $w(G^r) = G$ .

## An interlude on oligomorphy

$\text{Aut}(\mathbb{Q}; <)$  and  $\text{Sym}(\mathbb{N})$  are both oligomorphic:

### Definition (Oligomorphic)

$|\Omega| = \aleph_0$ .  $G \curvearrowright \Omega$  is **oligomorphic** if for all  $n \in \mathbb{N}$ ,  $G \curvearrowright \Omega^n$  has finitely many orbits in the action  $g \cdot (a_1, \dots, a_n) \mapsto (ga_1, \dots, ga_n)$ .

**Ryll-Nardzewski:**  $G = \text{Aut}(M)$  is oligomorphic  $\Leftrightarrow M$  is  $\omega$ -categorical.  
Many examples are **homogeneous**: every isomorphism between finite substructures of  $M$  extends to an automorphism.

## An interlude on oligomorphicity

Many examples are **homogeneous**: every isomorphism between finite substructures of  $M$  extends to an automorphism.

Fraïssé:

$$\left\{ \begin{array}{l} \text{classes of finite substructures (ages) of} \\ \text{homogeneous structures} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Fraïssé} \\ \text{classes} \end{array} \right\}.$$

**Fraïssé**:\* hereditary class  $\mathcal{C}$  with the **amalgamation property**:

(AP) for  $A, B_0, B_1 \in \mathcal{C}$ , embeddings  $f_i : A \rightarrow B_i$  ( $i \in \{0, 1\}$ ),  
 there is  $C \in \mathcal{C}$  and embeddings  $g_i : B_i \rightarrow C$  ( $i \in \{0, 1\}$ ) s.t.  
 $g_1 \circ f_1 = g_2 \circ f_2$ .

If language is finite,  $M$  is  $\omega$ -categorical.

Often, we specify homogeneous structures by their age:  
 random graph, generic triangle-free graph, random poset. . .

---

\*Assuming language is countable and purely relational.

# Mixed identities

## Definition (MIF and lawless)

$w(x_1, \dots, x_r)$  is a **mixed identity** if  $w(G^r) = 1$ .

A mixed identity without constants is a **law**.

$G$  is **mixed-identity-free (MIF)**: no (non-trivial) mixed identities.

$G$  is **lawless**: no (non-trivial) laws.

👁 Mixed identities describe the structure of a group/elements:

$$\forall x x^2 = 1 \quad \text{or} \quad \forall x xcx^{-1}c^{-1} = 1$$

💡 we would expect “rich group”  $\Rightarrow$  “few” mixed identities.

# Mixed identities

## Definition (MIF and lawless)

$w(x_1, \dots, x_r)$  is a **mixed identity** if  $w(G^r) = 1$ .

A mixed identity without constants is a **law**.

$G$  is **mixed-identity-free (MIF)**: no (non-trivial) mixed identities.

$G$  is **lawless**: no (non-trivial) laws.

## Theorem (Macpherson 1986)

$G \curvearrowright \Omega$  oligomorphic  $\Rightarrow G$  is lawless.

## Theorem (Melles and Shelah 1994)

$M$  a saturated structure<sup>†</sup>  $\Rightarrow \text{Aut}(M)$  is lawless.

---

<sup>†</sup>**saturated**: types over subsets  $A \subseteq M$  with  $|A| < |M|$  are realised in  $M$ .

## Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space  
(Etedadialiabadi, Gao, Le Maître, and Melleray 2021);
- transitive with free amalgamation except if  $\text{Aut}(M) = \text{Sym}(\mathbb{N})$   
(Ghadernezhad and de la Nuez González 2019; Bodirsky, Schneider, and Thom 2024);
- random poset, random permutation  
(Bodirsky, Schneider, and Thom 2024).

Sometimes we have mixed identities!

- $\text{Sym}(\mathbb{N})$ :  $[x, t]^6$  where  $t$  is a transposition  
(Hull and Osin 2016);
- $\text{GL}(\mathbb{N}_0, \mathbb{F}_q)$ :  
(Bradford, Schneider, and Thom 2023);
- $\text{Aut}(\mathbb{Q}; <)$ :  
(Bodirsky, Schneider, and Thom 2024).

# Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space  
(Etadialiabadi, Gao, Le Maître, and Melleray 2021);
- transitive with free amalgamation except if  $\text{Aut}(M) = \text{Sym}(\mathbb{N})$   
(Ghadernezhad and de la Nuez González 2019; Bodirsky, Schneider, and Thom 2024);
- random poset, random permutation  
(Bodirsky, Schneider, and Thom 2024).

In the first two cases, we know all mixed identities are singular!

- $\text{Sym}(\mathbb{N})$ : all mixed identities are singular  
(Bodirsky, Schneider, and Thom 2024);
  - $\text{GL}(\mathbb{N}_0, \mathbb{F}_q)$ : all mixed identities are singular  
(Bradford, Schneider, and Thom 2023);
  - $\text{Aut}(\mathbb{Q}; <)$ : ????
- (Bodirsky, Schneider, and Thom 2024).

## Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space ;
- transitive with free amalgamation except if  $\text{Aut}(M) = \text{Sym}(\mathbb{N})$  ;
- random poset, random permutation.
- $\text{Sym}(\mathbb{N})$ : all mixed identities are singular;
- $\text{GL}(\aleph_0, \mathbb{F}_q)$ : all mixed identities are singular ;
- $\text{Aut}(\mathbb{Q}; <)$ : ????.

Conjecture (Bodirsky, Schneider, and Thom 2024)

$G \curvearrowright \Omega$  oligomorphic  $\Rightarrow$  all mixed identities of  $G$  are singular.

We want general conditions that imply all mixed identities are singular.

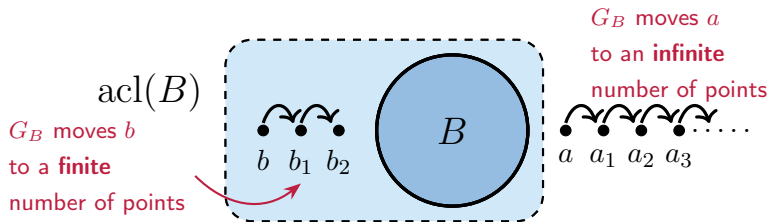
## Algebraic closure

## Definition (algebraic closure)

$G \curvearrowright \Omega$ .  $B \subseteq \Omega$  finite.

$$\text{acl}(B) := \{a \in \Omega \mid a \text{ has finite } G_B\text{-orbit}\}.$$

$G$  has **no algebraicity** if  $\text{acl}(B) = B$  for all finite  $B$ .



# Algebraic closure

## Definition (algebraic closure)

$G \curvearrowright \Omega$ .  $B \subseteq \Omega$  finite.

$$\text{acl}(B) := \{a \in \Omega \mid a \text{ has finite } G_B\text{-orbit}\} .$$

$G$  has **no algebraicity** if  $\text{acl}(B) = B$  for all finite  $B$ .

## Lemma (II. M. Neumann 1976)

$G \curvearrowright \Omega$ .  $A, B \subseteq \Omega$  finite and  $A \cap \text{acl}(\emptyset) = \emptyset$ .

*Then, there is  $g \in G$  such that  $(g \cdot A) \cap B = \emptyset$ .*

## Theorem (Abért 2005)

$G \curvearrowright \Omega$  with no algebraicity. Then,  $G$  is lawless.

# Our result

## Theorem (MP 2026)

$G \curvearrowright \Omega$  with no algebraicity.

Then, all mixed identities of  $G$  are singular.

Applies to plenty of groups:

### Examples (Examples from logic)

- automorphism groups of homogeneous structures whose age has the **strong amalgamation property**:
  - recovers most previous results; we also automatically cover reducts!
  - confirms the Bodirsky, Schneider, and Thom 2024 conjecture for several cases beyond previous techniques:
    - ( $\mathbb{Q}; <$ ) and reducts, Droste's 1985 2-homogeneous trees of negative type, homogeneous local order  $S(2)$ , homogeneous  $B, C, D$  relations; These have mixed identities (Ghadernzhad and de la Nuez González 2019);

# Our result

## Theorem (MP 2026)

$G \curvearrowright \Omega$  with no algebraicity.

Then, all mixed identities of  $G$  are singular.

Applies to plenty of groups:

### Examples (Examples from group theory)

- Thompson's groups  $F, T$ , and  $V$ ; Mixed identities studied in: (Zarzycki 2010; Śtanina and Zarzycki 2019; Ivanov and Zarzycki 2025; Elliott 2026).  
These have mixed identities. This is MIF;
- Weakly branch groups (include Grigorchuk's group).  
Has mixed identities (Jacobson 2021);
- Any wreath product  $G \wr H$  where  $G \curvearrowright \Omega$  has no algebraicity.  
Has mixed identities (Jacobson 2021);

## Our result

## Theorem (MP 2026)

$G \curvearrowright \Omega$  with no algebraicity.

Then, all mixed identities of  $G$  are singular.

Applies to plenty of groups:

## Examples (Examples from topology)

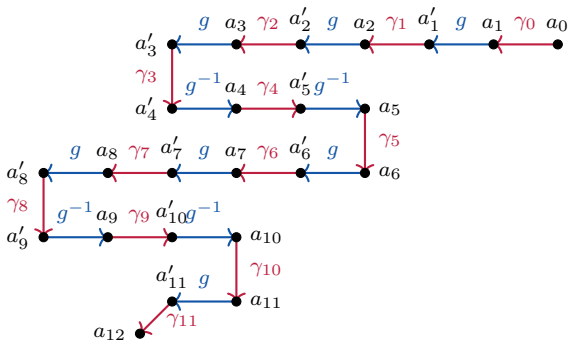
- Homeomorphism groups of: Cantor set  $2^\omega$ , Hilbert cube  $[0, 1]^\omega$ , any manifold of dimension  $\geq 1$ ; dimension  $\geq 2$  are MIF;  $\text{Homeo}(\mathbb{R}), \text{Homeo}(S^1)$  have mixed identities. (Elliott 2026);
- Diffeomorphism groups of smooth boundariless manifolds;
- Any group with a **Rubin action**;  
(Rubin 1989; Belk, Elliott, and Matucci 2025);
- $\text{Homeo}^+(0, 1), \text{Homeo}^+(S^1)$ ;  
(open problem in Bodirsky, Schneider, and Thom 2024);

Proof strategy<sup>‡</sup>

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n) .$$

$$w(x) = \gamma_{11}x\gamma_{10}x^{-1}\gamma_9x^{-1}\gamma_8x\gamma_7x\gamma_6x\gamma_5x^{-1}\gamma_4x^{-1}\gamma_3x\gamma_2x\gamma_1x\gamma_0$$



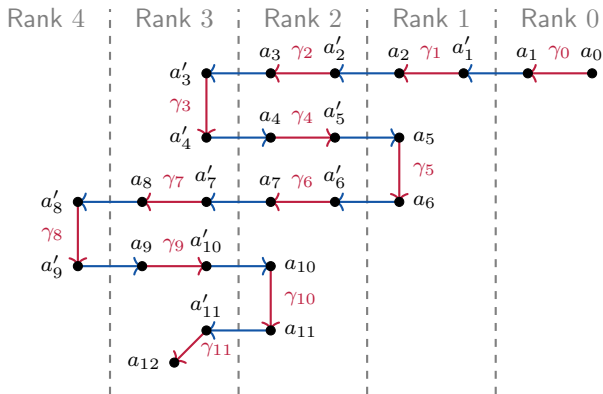
<sup>‡</sup> $w$  with one variable.

## Proof strategy

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n) .$$

Rank elements according to their position in  $\mathbb{Z}$ .

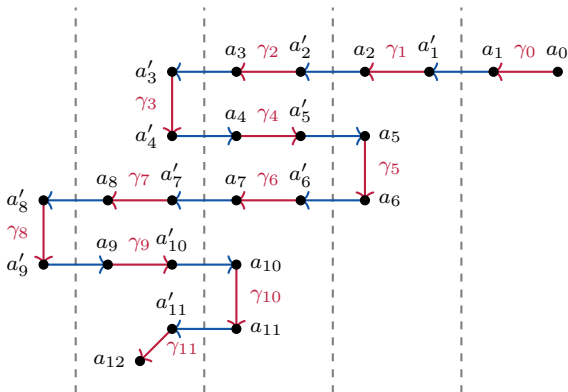


## Proof strategy

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n).$$

We want  $g$  so that elements of different ranks are independent!

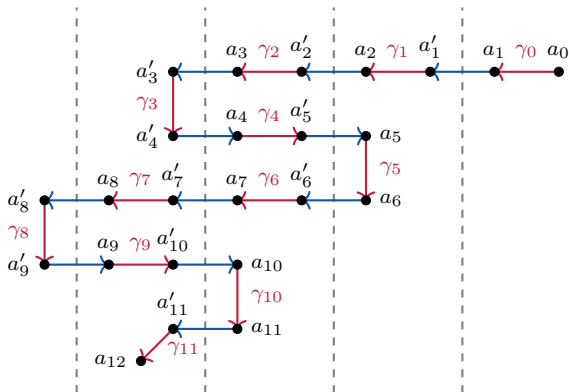


## Proof strategy

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n).$$

We cannot guarantee elements of same rank to be distinct!

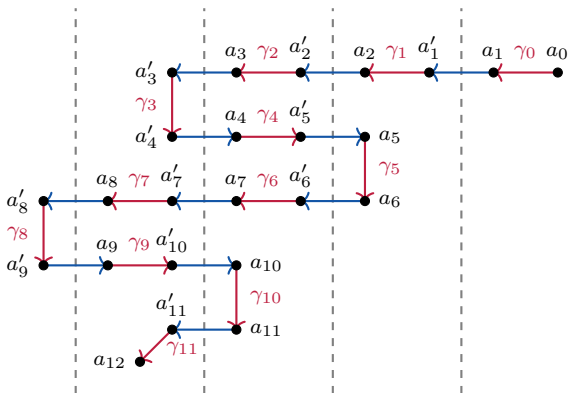


## Proof strategy

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n).$$

Use **Neumann's Lemma** to guarantee independence (looking ahead!)

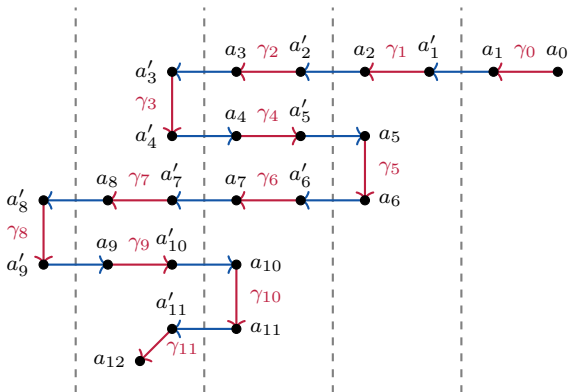


# Proof strategy

For  $g \in G$ ,  $a_0 \in \Omega$ , the images of  $a_0$  with respect to subwords of  $w$  create a sequence  $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$  with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n).$$

Induction proof: challenge is balancing inductive hypothesis!



## The real Theorem

With algebraicity, we need a further geometric properties and a strengthening of Neumann's Lemma.

### Definition

Let  $G \curvearrowright \Omega$ .  $(\Omega, \text{acl})$  is **higher Neumann** if for each  $B, C \subseteq \Omega$  finite,  $a \in \Omega$  such that  $\text{acl}(a) \cap \text{acl}(B) = \text{acl}(\emptyset)$ , and  $\Sigma \subseteq G$  finite,  $g \in G_B$  such that

$$\text{acl} \left( \bigcup_{\gamma \in \Sigma} \gamma g(a) \right) \cap \text{acl}(C) = \text{acl}(\emptyset) .$$

$\text{acl}$  forms a **pregeometry** if it satisfies:

(EXCHANGE) if  $a \in \text{acl}(A \cup \{b\}) \setminus \text{acl}(A)$ , then  $b \in \text{acl}(A \cup \{a\})$ .

It is **modular** if for all closed  $A, B \subseteq \Omega$ ,

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B) .$$

# The real Theorem

## Definition

Let  $G \curvearrowright \Omega$ .  $(\Omega, \text{acl})$  is **higher Neumann** if for each  $B, C \subseteq \Omega$  finite,  $a \in \Omega$  such that  $\text{acl}(a) \cap \text{acl}(B) = \text{acl}(\emptyset)$ , and  $\Sigma \subseteq G$  finite,  $g \in G_B$  such that

$$\text{acl} \left( \bigcup_{\gamma \in \Sigma} \gamma g(a) \right) \cap \text{acl}(C) = \text{acl}(\emptyset) .$$

## Theorem (MP 2026)

*Let  $G \curvearrowright \Omega$  be s.t.  $\text{acl}$  forms an infinite-dimensional higher Neumann modular pregeometry. Then all mixed identities of  $G$  are singular.*





Applies to  $\text{GL}(\aleph_0, \mathbb{F}_q)$  and  $\text{PGL}(\aleph_0, \mathbb{F}_q)$ , recovering results of Bradford, Schneider, and Thom 2023.

Thank you!





Recap:

- Mixed identities are a way to describe structure in groups;
- We prove: all mixed identities are singular for groups with no algebraicity;
- Theorem covers many classes of groups for which people are studying their mixed identities.




# Bibliografy I

-  Abért, Miklós (2005). “Group laws and free subgroups in topological groups”. In: *Bulletin of the London Mathematical Society* 37.4, pp. 525–534.
-  Adeleke, Samson A and W Charles Holland (1994). “Representation of order automorphisms by words”. In.
-  Banakh, Taras, Igor Guran, and Igor Protasov (2012). “Algebraically determined topologies on permutation groups”. In: *Topology and its Applications* 159.9, pp. 2258–2268.
-  Belk, James, Luna Elliott, and Francesco Matucci (2025). “A short proof of Rubin’s theorem”. In: *Israel Journal of Mathematics* 267.1, pp. 157–169.
-  Bodirsky, Manuel, Jakob Schneider, and Andreas Thom (2024). “Mixed identities for oligomorphic automorphism groups”. In: *arXiv preprint arXiv:2401.09205*.



## Bibliografy II

-  Bradford, Henry, Jakob Schneider, and Andreas Thom (2023). “Non-singular word maps for linear groups”. In: *arXiv preprint arXiv:2311.03981*.
-  De la Nuez Gonzalez, J., Zaniar Ghadernezhad, Paolo Marimon, and Michael Pinsky (2026). *Minimal and intrinsic topologies on monoids of elementary embeddings*. arXiv: 2603.28419 [math.LO]. URL: <https://arxiv.org/abs/2603.28419>.
-  Droste, Manfred (1985). *Structure of partially ordered sets with transitive automorphism groups*. American Mathematical Society.
-  Elliott, Luna (2026). *The Zariski Topology on Homeomorphism groups*. arXiv: 2601.15185 [math.GR]. URL: <https://arxiv.org/abs/2601.15185>.







## Bibliografy III

-  Etedadialiabadi, Mahmood, Su Gao, François Le Maître, and Julien Melleray (2021). “Dense locally finite subgroups of automorphism groups of ultraextensive spaces”. In: *Advances in Mathematics* 391, p. 107966.
-  Gerstenhaber, Murray and Oscar S Rothaus (1962). “The solution of sets of equations in groups”. In: *Proceedings of the National Academy of Sciences* 48.9, pp. 1531–1533.
-  Ghadernezhad, Zaniar and J. de la Nuez González (2019). “Group topologies on automorphism groups of homogeneous structures”. In: *arXiv e-prints*. Note: the ArXiv version of this paper contains results on mixed identities that are not in the published version, and which are submitted for publication for a paper with different title., arXiv–1909.





## Bibliografy IV

-  Hull, Michael and Denis Osin (2016). “Transitivity degrees of countable groups and acylindrical hyperbolicity”. In: *Israel Journal of Mathematics* 216.1, pp. 307–353.
-  Ivanov, Aleksander and Roland Zarzycki (2025). “Mixed identities, hereditarily separated actions and oscillation”. In: *Journal of Group Theory* 28.3, pp. 585–623.
-  Jacobson, Bryan (2021). “A mixed identity-free elementary amenable group”. In: *Communications in Algebra* 49.1, pp. 235–241.
-  Kervaire, Michel A (2015). “On higher dimensional knots”. In: *Differential and combinatorial topology*. Princeton University Press, pp. 105–120.
-  Klyachko, Anton and Andreas Thom (2017). “New topological methods to solve equations over groups”. In: *Algebraic & Geometric Topology* 17.1, pp. 331–353.

## Bibliografy V

-  Lyndon, Roger C (1990). “Words and infinite permutations”. In: *Mots. Langue, Raisonement, Calcul, Hermes, Paris*, pp. 143–152.
-  Macpherson, H Dugald (1986). “Groups of automorphisms of  $\aleph_0$ -categorical structures”. In: *The Quarterly Journal of Mathematics* 37.4, pp. 449–465.
-  Melles, Garvin and Saharon Shelah (1994). “ $\text{Aut}(M)$  has a large dense free subgroup for saturated  $M$ ”. In: *Bulletin of the London Mathematical Society* 26.4, pp. 339–344.
-  Mycielski, Jan (1987). “Representations of infinite permutations by words”. In: *Proceedings of the American Mathematical Society* 100.2, pp. 237–241.
-  Neumann, Peter M (1976). “The structure of finitary permutation groups”. In: *Archiv der Mathematik* 27.1, pp. 3–17.
-  Pestov, Vladimir G (2008). “Hyperlinear and sofic groups: a brief guide”. In: *Bulletin of Symbolic Logic* 14.4, pp. 449–480.

## Bibliografy VI

-  Pinsker, Michael and Clemens Schindler (2026). “On the Zariski topology on endomorphism monoids of omega-categorical structures”. In: *The Journal of Symbolic Logic* 91.1, pp. 317–335.
-  Rubin, Matatyahu (1989). “On the reconstruction of topological spaces from their groups of homeomorphisms”. In: *Transactions of the American Mathematical Society* 312.2, pp. 487–538.
-  Słanina, Piotr and Roland Zarzycki (2019). “Laws with constants in Thompson’s group  $F$ ”. In: *Journal of Group Theory* 22.5, pp. 783–793.
-  Zarzycki, Roland (2010). “Limits of Thompson’s group  $F$ ”. In: *Combinatorial and Geometric Group Theory: Dortmund and Ottawa-Montreal Conferences*. Springer, pp. 307–315.

## Extrinsic and intrinsic topologies

Our original motivation comes from studying topologies on groups/semigroups:

### Definition (Topology of pointwise convergence)

$G \curvearrowright \Omega$  has a natural **pointwise convergence topology**  $\tau_{\text{pw}}$  induced by the product topology on  $\Omega^\Omega$  ( $\Omega$  discrete). Sub-basis of open sets:

$$\mathcal{U}_{(a,b)} := \{g \in G \mid g(a) = b\},$$

for  $a, b \in \Omega$ .

Our Theorem shows that solution sets to regular words are nowhere dense in  $G^r$ .

## Extrinsic and intrinsic topologies

### Definition (Topology of pointwise convergence)

$G \curvearrowright \Omega$  has a natural **pointwise convergence topology**  $\tau_{\text{pw}}$  induced by the product topology on  $\Omega^\Omega$  ( $\Omega$  discrete). Sub-basis of open sets:

$$\mathcal{U}_{(a,b)} := \{g \in G \mid g(a) = b\} ,$$

for  $a, b \in \Omega$ .

### Definition (Zariski topologies)

The **group Zariski topology**  $\tau_Z^G$  on  $G$  has as a sub-basis of open sets the solution sets to mixed disequalities in one variable:

$$\{g \in G \mid w(g) \neq 1\} .$$

The **semigroup Zariski topology** on a semigroup  $S$   $\tau_Z^S$  is defined analogously with solution sets of semigroup disequalities:

$$\{s \in S \mid w(g) \neq v(g)\} ,$$

where  $w, v$  are mixed semigroup words. (no inverses!)

## What can $\tau_Z$ describe?

- $\tau_Z^G$  is contained in any Hausdorff group topology on  $G$ ;
- $\tau_Z^S$  is contained in any Hausdorff semigroup topology on  $S$ ;
- they need not be Hausdorff or group/semigroup topologies!

### Question

What is  $\tau_Z$  for our favorite groups/semigroups?

We know  $\tau_{\text{pw}} = \tau_Z^G$  for:

- $\text{Sym}(\mathbb{N})$  Banakh, Guran, and Protasov 2012;
- $\text{Aut}(\mathbb{Q}; <)$  Ghadernezhad and de la Nuez González 2019.

In topological cases,  $\tau_Z^G$  can describe other interesting topologies:

- In Thompson's groups  $F$  and  $T$ , and  $\text{Homeo}(\mathbb{R})$  and  $\text{Homeo}(S^1)$ ,  $\tau_Z^G$  is the **compact-open topology**  $\tau_{\text{co}}$ :  
for  $X$  a topological space, sub-basis of open sets

$$\mathcal{U}_{K,U} := \{f \in \text{Homeo}(X) \mid f(K) \subseteq U\},$$

for  $K$  compact and  $U$  open. (Elliott 2026).

# Zariski and MIF

## Definition

A topological space  $X$  is **irreducible** if whenever  $K_1, K_2 \subsetneq X$  are closed sets, we have  $K_1 \cup K_2 \neq X$ .

## Proposition (Elliott 2026)

*Let  $G$  be MIF. Then  $\tau_Z^G$  is irreducible (and so not Hausdorff).*

We can use results from the literature showing  $\tau_Z^G$  is Hausdorff to deduce the existence of mixed identities.

## Some results on semigroups

What about  $\tau_Z^S$  on  $\overline{G}$ ?

For  $M$   $\omega$ -categorical,  $\overline{G} = \text{EEmb}(M)$ .

## Some results on semigroups

Theorem (Pinsker and Schindler 2026)

$M$   $\omega$ -categorical with no algebraicity. Then,  $\tau_Z^S = \tau_{pw}$  on  $\text{EEmb}(M)$ .

Theorem (de la Nuez Gonzalez, Ghadernezhad, Marimon, and Pinsker 2026)

$G \curvearrowright \Omega$  has locally finite  $\text{acl}^\dagger$ . Let  $w, v$  be semigroup words (in one variable) and  $w(x) = v(x)$  be regular. Then, for  $S \in \{G, \overline{G}\}$ ,  

$$\{s \in S \mid w(s) = v(s)\}$$
 is nowhere dense in  $S$ .

💡 all semigroup mixed identities are singular!

---

$\dagger$ For  $B$  finite,  $\text{acl}(B)$  finite.

## Some results on semigroups

Theorem (Pinsker and Schindler 2026)

$M$   $\omega$ -categorical with no algebraicity. Then,  $\tau_Z^S = \tau_{pw}$  on  $\text{EEmb}(M)$ .

Theorem (de la Nuez Gonzalez, Ghadernezhad, Marimon, and Pinsker 2026)

$G \curvearrowright \Omega$  has locally finite  $\text{acl}^\dagger$ . Let  $w, v$  be semigroup words (in one variable) and  $w(x) = v(x)$  be regular. Then, for  $S \in \{G, \overline{G}\}$ ,

$$\{s \in S \mid w(s) = v(s)\}$$

is nowhere dense in  $S$ .

Corollary (de la Nuez Gonzalez, Ghadernezhad, Marimon, and Pinsker 2026)

With the above assumptions+  $Z(G) \supsetneq \{1\}$ ,  
 $\tau_Z^S$  is not Hausdorff on  $S$ .

---

$\dagger$ For  $B$  finite,  $\text{acl}(B)$  finite.