The Tree Interpretation and Transfinite Recursion

(joint work with Emanuele Frittaion)

Giorgio G. Genovesi
University of Leeds, School of Mathematics
mmggg@leeds.ac.uk

February 25, 2026

Definitions and motivation

Classic result

$$\mathbf{Z}_2 \rhd \mathbf{ZFC}^-$$

Where $\mathbf{Z}_2$ is second order arithmetic with the full comprehension scheme and $\mathbf{ZFC}^-$ is $\mathbf{ZFC}$ without the powerset axiom.

$$\mathbf{Z}_2 \rhd \mathbf{ATR}_0^{set} + \Pi^1_\infty \text{ Separation } \rhd \mathbf{ZFC}^-$$

$\mathbf{ATR}_0^{set}$ is a theory of sets with Axiom β or the Mostowki Collapse Lemma.

1. First interpretation is with well founded trees with an appropriate relation that ensures extensionality. This corresponds to the natural surjection $\mathcal{P}(\omega) \to H(\aleph_1)$.
2. The second interpretation is the restriction to an initial segment of L.

Examples of such interpretations

1. Zbierski [1] to show that $\mathbf{Z}_{n+2} \rhd \mathbf{ZFC}^- + \mathcal{P}^n(\omega)$ exists.
2. Sochor [2] gives a direct construction of L in $\mathbf{Z}_{n+2}$ to show the equiconsistency with choice.
3. Kanovei [3] showed that $\mathbf{Z}^- \rhd \mathbf{ZFC}^-$.
4. The interpretation which considers all relations (not just well founded) modulo an appropriate bisimulation can be found in Aczel's book [4].
5. Mathias [5] also does this two step intepretation in Mac Lane set theory.
6. Simpson [6] [7] isolates a minimal subsystem of $\mathbf{Z}_2$ needed for the the tree interpretation and the construction of L.

Collapse Functions and Bisimulations

Let R be a relation on x, we say $B \subseteq x \times x$ is a bisimulation for R if aBc then

$$(\forall uRa\ \exists vRc\ (uBv)) \wedge (\forall vRc\ \exists uRa\ (uBv))$$

A collapse function for R is a map $\pi : x \to y$, where y is transitive and

$$\pi(u) = \{\pi(v) : vRu\}$$

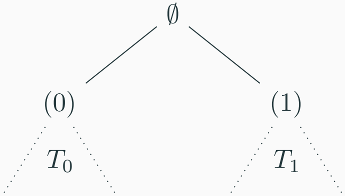
Note that the fibers of a collapse function will be a bisimulation by the axiom of extensionality.

We will mostly consider trees $T \subseteq X^{<\omega}$ with the relation of immediate predecessor.

The Tree interpretation

Definition
$Pair(T_0, T_1) = \{\emptyset\} \cup (0)^\frown T_0 \cup (1)^\frown T_1$

$Pair(T_0, T_1) \quad =$



The tree interpretation will have as domain the class of all well founded trees.

- $T_0 =^* T_1 \equiv \langle (0), (1) \rangle \in B$ a bisimulation on $Pair(T_0, T_1)$.
- $T_0 \in^* T_1 \equiv \exists \sigma \in T_1 \ |\sigma| = 1 \wedge T_0 =^* \{\tau : \sigma^\frown \tau \in T_1\}$.

Systems of reverse math

Work in the language of second order arithmetic $\mathcal{L}_2 = \{0, S, +, \cdot, <, \in\}$, with only first order equality as a symbol, and Henkin semantics.

- $\mathbf{RCA}_0 \equiv \mathbf{Q} + \mathbf{I}\Sigma^0_1 + \Delta^0_1$ Comprehension.
- $\mathbf{WKL}_0 \equiv \mathbf{RCA}_0 +$ Weak König's lemma.
- $\mathbf{ACA}_0 \equiv \mathbf{RCA}_0 +$ Arithmetic Comprehension.
- $\mathbf{ATR}_0 \equiv \mathbf{RCA}_0 +$ Arithmetic Transfinite Recursion.
- $\Pi^1_1\text{-}\mathbf{CA}_0 \equiv \mathbf{RCA}_0 +$ Comprehension for $\forall X \varphi(n, X)$ formulas.
- $\mathbf{Z}_2 \equiv \mathbf{RCA}_0 +$ Full Comprehension

Systems of set theory

1. Extensionality.
2. Pair.
3. Union.
4. Δ_0 Separation.
5. Infinity.
6. Transitive Closure.
7. Regularity.
8. Finite Powerset.
9. Axiom β (every well founded relation has a collapse function).
10. Countability.

$\mathbf{B}$ is the system with axiom 1-8, $\mathbf{C}$ is 1-9, and $\mathbf{ATR}_0^{set}$ is all the above.

The Tree interpretation

1. $\mathbf{RCA}_0$+ Every well founded relation has a bisimulation is equivalent to $\mathbf{ATR}_0$.
2. $\mathbf{ATR}_0 \rhd \mathbf{ATR}_0^{set}$, in fact the two systems are biinterpretable.
3. $\mathbf{Z}_2 \rhd \mathbf{ATR}_0^{set}$+ Full Separation but won't interpret Σ_2 Replacement!
4. $\mathbf{ATR}_0^{set} \vdash \forall\alpha \in Ord\,(L_\alpha$ exists).
5. $\mathbf{ATR}_0^{set} \nvdash (\mathbf{ATR}_0^{set})^L$, in fact $\mathbf{ATR}_0^{set} + V = L \vdash Con(\mathbf{ATR}_0^{set})$.
6. $\mathbf{ATR}_0^{set} \vdash (\mathbf{KP}^r)^{L_{\omega_1^{CK}}}$
7. $\mathbf{ATR}_0$ can actually prove the existence of an ω-model of $\mathbf{KP}^r + \Pi_1$-Foundation.
8. $\mathbf{ATR}_0^{set}$ does not have a minimal transitive model!
9. $\mathbf{ATR}_0 + \Pi^1_{n+1}$ Comprehension $\rhd \mathbf{ATR}_0^{set} + \Sigma_n$ Separation.

Generalizing these results

Given a $\{\in\}$-theory $\mathbf{T}$ by $\mathbf{T}^\gamma$ we mean the theory $\mathbf{T}$ with the axiom

$$\gamma \in Ord \wedge \mathcal{P}^\gamma(\omega) \text{ exists}$$

Where $\mathcal{P}^0(\omega) = \omega$ and $\mathcal{P}^\gamma(\omega) = \bigcup_{\beta<\gamma} \mathcal{P}(\mathcal{P}^\beta(\omega))$. For such systems it is natural to consider interpretations that preserve γ ($\mathbf{T}^\gamma \rhd_\gamma \mathbf{S}^\gamma$).

The system $\mathbf{B}^n$ corresponds roughly to arithmetic of $n+2$ order. In fact the two systems are mutually interpretable.

We may consider the tree interpretation with well founded $T \subseteq (\mathcal{P}^\gamma(\omega))^{<\omega}$. Start with

$$\mathbf{B}^\gamma + \Delta_0\text{-}\mathbf{TR} \rhd_\gamma \mathbf{C}^\gamma + \Delta_0\text{-}\mathbf{TR}$$

And in $\mathbf{C}^\gamma + \Delta_0\text{-}\mathbf{TR}$ one can construction L just like in Simpson.

Note $\mathbf{B}^\gamma + \Delta_0\text{-}\mathbf{TR} \subseteq \mathbf{B}^\gamma + \Sigma_1$ Separation.

Interpretation results

By the the tree interpretation and restriction in L we get

1. When $\gamma = 0$ or $cf(\gamma) = \omega$
   1. $\mathbf{B}^\gamma + \Sigma_{k+2}$ Separation $\rhd_\gamma$ $\mathbf{C}^\gamma + \Sigma_{k+1}$ Strong Collection $+ V = L$.
   2. $\mathbf{B}^\gamma + \Delta_{k+2}$ Separation $\rhd_\gamma$ $\mathbf{C}^\gamma + \Sigma_{k+1}$ Collection $+ V = L$
2. When $\gamma = \beta + 1$ or $cf(\gamma) > \omega$
   1. $\mathbf{B}^\gamma + \Sigma_{k+1}$ Separation $\rhd_\gamma$ $\mathbf{C}^\gamma + \Sigma_{k+1}$ Strong Collection $+ V = L$.
   2. $\mathbf{B}^\gamma + \Delta_{k+1}$ Separation $\rhd_\gamma$ $\mathbf{C}^\gamma + \Sigma_{k+1}$ Collection $+ V = L$

If $\gamma = 0$ or $cf(\gamma) = \omega$ we have $|\mathcal{P}^{\gamma+1}(\omega)| \leq^* |[\mathcal{P}^\gamma(\omega)]^\omega|$.

If γ is a successor or $cf(\gamma) > \omega$ then $|\mathcal{P}^\gamma(\omega)| = |[\mathcal{P}^\gamma(\omega)]^\omega|$.

This difference means in the first case being well founded is Δ_0 relative to $\mathcal{P}^\gamma(\omega)$ while in the second case it is Π^1_1 relative to $\mathcal{P}^\gamma(\omega)$. In fact some form of the Kleene normal form holds.

Limits

In the case $\gamma = 0 \vee cf(\gamma) = \omega$ we cannot do better!

$$\mathbf{C}^\gamma + \Sigma_{k+1} \text{ Separation} \rhd \mathbf{B}^\gamma + \Sigma_{k+2} \text{ Separation}$$

We also get the following results.

$$\mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation} \vdash Con(\mathbf{A}^\gamma + \Sigma_k \text{ Separation })$$

so in particular

$$\mathbf{A}^\gamma + \Sigma_k \text{ Separation} \quad \not\rhd \mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation}$$

For γ successor and uncountable cofinality we have

$$\mathbf{C}^\gamma + \Sigma_k \text{ Separation } + \Sigma_k \text{ Collection } \not\rhd \mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation}$$

If $\beta < \gamma$ then

$$\mathbf{A}^\gamma + \Delta_0 \text{ Fixed Point} \vdash Con(\mathbf{ZF}^- + \mathcal{P}^\beta(\omega) \text{ exists})$$

Δ_{n+1} Separation

The systems $\mathbf{A}^\gamma + \Delta_{n+1}$ Separation without any additional induction turns out to be conservative over $\mathbf{A}^\gamma + \Sigma_n$ separation.

Theorem (McKenzie [8])
$\mathbf{KP}^r + \Sigma_{k+2}$ Collection $+ V = L$ is Π_{k+3} conservative over $\mathbf{KP}^r + \Sigma_{k+1}$ Separation $+ \Sigma_{k+1}$ Collection $+ V = L$.

The following proof makes use of cuts which can be arbitrarily high in a model of $\mathbf{KP} + V = L$ so analogous results for $\mathbf{C}^\gamma$ hold. Pairing this with Shoenfield Absolutness it follows that $\mathbf{A} + \Delta_{n+1}$ Separation is Σ^1_2 conservative over $\mathbf{A} + \Sigma_n$ Separation. With more induction on ω we get a different picture.

Theorem (McKenzie[8])
$\mathbf{KP}^r + \Sigma_{k+2}$ Collection $+ \Sigma_{k+2}$ Induction proves the consistency of $\mathbf{KP}^r + \Sigma_{k+1}$ Separation $+ \Sigma_{k+1}$ Collection $+$ Foundation.

Similar results are obtained for $\mathbf{C}^\gamma$.

The case with Δ_0-Separation

In the case for Δ_0 separation it is not clear if the following even holds

1. $\mathbf{B}^\gamma \rhd \mathbf{C}^\gamma$?
2. Can $\mathbf{C}$ prove $\forall \alpha \in Ord\,(L_\alpha$ exists$)$?
3. Is Δ_0-$\mathbf{TR}$ necessary?

It turns out that over $\mathbf{B}$ the following are equivalent

1. Every well founded tree has a bisimulation.
2. Δ_0-$\mathbf{TR}$.
3. Clopen Denterminacy.

The equivalence of 2 and 3 was proven by Gitman and Hamkins [9] over $\mathbf{NBG}$.

$\mathbf{C}$ proves both Δ_0-$\mathbf{TR}$ and $\forall b \forall \alpha \in Ord\,(L_\alpha(b)$ exists $)$.

On the finite powerset

Mathias [10] shows that adding the finite powerset axiom fixes many of the issues predicative set theories have. Even with axiom β the lack of the finite powerset axiom will cause issues.

It is consistent with $\mathbf{ZF}$ that there exists and $A \subseteq \mathcal{P}(\omega)$ such that

$$\aleph_0 <^* |A| <^* |A^2| <^* |A^3| <^* \cdots <^* |A^{<\omega}|$$

In fact the set A in the Basic Cohen model will be dual Dedekind finite and by modifying Truss' argument you can show that for all $n \in \omega$ A^n is dual Dedekind finite.

In such model the set $\{x : \exists n \in \omega\ |TC(x)| \leq^* |A^n|\}$ will be a model of $\mathbf{C}$ with the finite powerset axiom replace by the closure under the Cartesian product.

An alternative interpretation

To carry out the $*$ interpretation we need to be able to code sequences.

1. $\mathbf{A}^\gamma$ proves that there is Δ_0 definable coding for sequence of $\mathcal{P}^\gamma(\omega)$ with elements of $\mathcal{P}^\gamma(\omega)$
2. However, $\mathbf{A}^\gamma$ does not prove the existence of bisimulation for all well founded trees of coded sequences.
3. We can however restrict ourselves to a smaller class of well founded trees to which $\mathbf{A}^\gamma$ can prove the existence of bisimulations.
4. This gives rise to the $\mathbf{b}*$ translation, which has the advantage that it preserves formula complexity.

We have $k \geq 0$

$$\mathbf{A}^\gamma + \Sigma_k \text{ Separation} \rhd \mathbf{B}^\gamma + \Sigma_k \text{ Separation}$$

The definition of the trees in the $\mathbf{b}*$ translation is technical but it captures the fact that the trees corresponding to the transitive closure and finite powerset have lots of repetitions.

Interpreting B

Let $n, k \in \omega \cup \{\infty\}$

$$\mathsf{ACA}_0 + \Sigma^1_n\text{-}\mathsf{CA}_0 + \Sigma^1_k\text{-}\mathsf{IND} \rhd \mathbf{B} + \Sigma_n \text{ Separation} + \Sigma_k \text{ Foundation}$$

Using the same translation one can get more correspondences between extensions of $\mathbf{ACA}_0$ and $\mathbf{B}$, for example

$$\mathbf{ATR}_0 \rhd \mathbf{B} + \Delta_1 \text{ Separation} + \Delta_0 \text{ Fixed Point}$$

even interpretations for theories of classes, for example

$$\mathbf{NBG} \rhd \mathbf{B}^\gamma + \gamma \text{ is inaccessible}$$

In such interpretations $\mathcal{P}^\gamma(\omega)$ is preserved, so the theories on the right hand side are $\mathcal{L}_2$ conservative extensions.

References

[1] P. Zbierski. **"Models for higher order arithmetics".** In: *Bull. Acad. Polon. Sci.* 19.7 (1971), pp. 557–562.

[2] A. Sochor. **"Constructibility in higher order arithmetics".** In: *Arch. Math. Logic* 32.6 (1993), pp. 381–389. ISSN: 0933-5846,1432-0665. DOI: 10.1007/BF01270463. URL: https://doi.org/10.1007/BF01270463.

[3] V. G. Kanovei. **"Theory of Zermelo without power set axiom and the theory of Zermelo-Fraenkel without power set axiom are relatively consistent".** English. In: *Math. Notes* 30 (1981), pp. 695–702. ISSN: 0001-4346. DOI: 10.1007/BF01141627.

[4] Peter Aczel. **Non-Well-Founded Sets.** Palo Alto, CA, USA: Csli Lecture Notes, 1988.

[5] A.R.D. Mathias. **"The strength of Mac Lane set theory".** In: *Annals of Pure and Applied Logic* 110.1 (2001), pp. 107–234. ISSN: 0168-0072. DOI: https://doi.org/10.1016/S0168-0072(00)00031-2.

[6] S. G. Simpson. **"Set Theoretic Aspects of ATR_0".** In: *Logic Colloquium '80*. Ed. by D. Van Dalen, D. Lascar, and T.J. Smiley. Vol. 108. Studies in Logic and the Foundations of Mathematics. Elsevier, 1982, pp. 255–271. DOI: https://doi.org/10.1016/S0049-237X(09)70515-4. URL: https://www.sciencedirect.com/science/article/pii/S0049237X09705154.

[7] S. G. Simpson. **Subsystems of Second Order Arithmetic.** 2nd. Perspectives in Logic. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511581007.

[8] Zachiri McKenzie. **"On the relative strengths of fragments of collection".** In: *Math. Log. Q.* 65.1 (2019), pp. 80–94. DOI: 10.1002/MALQ.201800044. URL: https://doi.org/10.1002/malq.201800044.

[9] Victoria Gitman and Joel David Hamkins. **"Open determinacy for class games".** In: AMS, Jan. 2017, pp. 121–143. ISBN: 9781470422561. DOI: 10.1090/conm/690/13865.

[10] A. R. D. Mathias. **"Weak Systems of Gandy, Jensen and Devlin".** In: *Set Theory: Centre de Recerca Matemàtica Barcelona, 2003–2004*. Basel: Birkhäuser Basel, 2006, pp. 149–224. ISBN: 978-3-7643-7692-5. DOI: 10.1007/3-7643-7692-9_6. URL: https://doi.org/10.1007/3-7643-7692-9_6.

Thank you for your attention!