Index Sets and Computable Categoricity of CSC Spaces

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Motivation

Question

Given a topological space X, how hard is it to describe?

Two approaches:

1. **Index Sets:** Assign an index to each "computable topological space" and locate

$$\{e \in \omega : X \text{ is homeomorphic to the space with index } e\}$$

in the arithmetic hierarchy.

2. **Computable Structure Theory:** Given two computable presentations of X, does there exist a computable homeomorphism between them? If not, how many computable copies of X exist up to computable homeomorphism?

Polish Metric Spaces

These types of questions have been studied (see e.g. Thewmorakot 2023) for Polish (complete and separable) metric spaces.

Index Sets (Thewmorakot 2023)

- $\{e : e$ is an index for a Polish space$\}$ is Π^0_2-complete.
- $\{e : e$ is an index for a discrete Polish space$\}$ is Π^1_1-complete.
- $\{e : e$ is an index for a perfect Polish space$\}$ is Π^0_2-complete.

"Computable isomorphisms" for Polish spaces are computable isometries.

Categoricity

- $2^{\mathbb{N}}$ is computably categorical (Melnikov 2013).
- $\mathbb{N}^{\mathbb{N}}$ is computably categorical (Thewmorakot 2023).
- $C[0, 1]$ is not computably categorical (Melnikov 2013).

CSC Spaces

Definition (Dorais 2011)

A **countable second-countable space** (**CSC space**) is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U} = (U_i)_{i \in \omega}$ is a countable basis for open sets in X, and k is a function $X \times \omega \times \omega \to \omega$ such that

- for all $x \in X$, there is $i \in \omega$ such that $x \in U_i$,
- for all $x \in X$ and $i, j \in \omega$, if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

CSC spaces provide an excellent context for studying topological facts in computability theory and reverse mathematics (Dorais 2011, Shafer 2020, Benham et al. 2024).

Three Motivating Examples

Example

Let $\mathbb{N}_{IND}$ be the CSC space $(\omega, \mathcal{U}, k)$ with $U_i = \omega$ for all i, and $k(x, i, j) = i$. Then $\mathbb{N}_{IND}$ has the **indiscrete topology**.

Example

Let $\mathbb{N}_{DIS}$ be the CSC space $(\omega, \mathcal{U}, k)$ with $U_i = \{i\}$ for all i, and $k(x, i, j) = i$. Then $\mathbb{N}_{DIS}$ has the **discrete topology**.

Example

Let $\mathbb{N}_{IST}$ be the CSC space $(\omega, \mathcal{U}, k)$ with $U_i = [0, i]$ for all i, and $k(x, i, j) = \min(i, j)$. Then $\mathbb{N}_{IST}$ has the **initial segment topology**.

Outline

Index Sets

Definition

A CSC space $(\omega, \mathcal{U}, k)$ is **computable** if $\mathcal{U}$ is uniformly computable and k is computable. That is, there are indices m and n such that Φ_m and Φ_n are total, $k = \Phi_n$, and

$$x \in U_i \iff \Phi_m(i, x) = 1$$

for all $i, x \in \omega$.

Then $\langle m, n \rangle$ is an **index** for a CSC space. Write

$$CSC = \{e : e \text{ is an index for a CSC space}\}.$$

Theorem

The set CSC is Π^0_2-complete.

Many-One Reductions

Definition

A set B is **many-one reducible** to a set A, written $B \leq_m A$, if there is a computable function f such that

$$x \in B \iff f(x) \in A$$

for all $x \in \omega$.

Definition

Let Γ be a complexity class.

- A set A is **Γ-hard** if $B \leq_m A$ for all $B \in \Gamma$.
- A set A is **Γ-complete** if $A \in \Gamma$ and A is Γ-hard.

Index Sets

Our goal is to classify the following sets:

$$IND = \{e : e \text{ is an index for a CSC space homeomorphic to } \mathbb{N}_{IND}\}$$
$$DIS = \{e : e \text{ is an index for a CSC space homeomorphic to } \mathbb{N}_{DIS}\}$$
$$IST = \{e : e \text{ is an index for a CSC space homeomorphic to } \mathbb{N}_{IST}\}$$

We classify their complexity **as subsets of** CSC.

Many-One Reductions *within* a Set

Definition (Calvert 2005, Knight)

Let Γ be a complexity class, let I be a set, and let A be a set.

- The set A is **Γ-within** I if $A = B \cap I$ for some $B \in \Gamma$.
- The set A is **Γ-hard within** I if for every $B \in \Gamma$, there is a computable function f such that
$$x \in B \iff f(x) \in A$$
and $f(x) \in I$, for all $x \in \omega$.
- The set A is **Γ-complete within** I if A is Γ-within I and Γ-hard within I.

Idea

Use $I = CSC$ and $A = IND$, DIS, and IST.

Index Sets

Strategy

Let Γ be a complexity class, and let $A \subseteq CSC$ be an index set of CSC spaces with some desired property. To show A is Γ-hard within CSC:

1. Fix a set $B \in \Gamma$, and let $e \in \omega$.
2. Define a sequence $\mathcal{V}^e = (V_i^e)_{i \in \omega}$ of subsets of ω uniformly computable in e.
3. Close $\mathcal{V}^e$ under finite intersection via primitive recursion (see Dorais 2011) to get a CSC space $X_e = (\omega, \mathcal{U}, k)$ with index $\langle m(e), n(e) \rangle$ uniformly computable in e.
4. Show X_e has the desired property if and only if $e \in B$. (It follows that $\langle m(e), n(e) \rangle \in A$ if and only if $e \in B$.)

Index Set Results

Theorem

IND is Π^0_1-complete within CSC.

Theorem

DIS is Π^0_3-complete within CSC.

Theorem

IST is Π^0_3-complete within CSC.

Index Set Results

Theorem

DIS is Π^0_3-complete within CSC.

Proof sketch.

A CSC space $(X, \mathcal{U}, k)$ is discrete if and only if

$$\forall x \exists i \forall y (y \in U_i \longleftrightarrow x = y).$$

Fix $e \in \omega$. The CSC space X generated by

$$V_{\langle x,y \rangle} = \begin{cases} \{x\} \cup \{s : \Phi_{e,s}(y) \downarrow\} & \text{if } y \geq x \\ \omega & \text{otherwise} \end{cases}$$

is discrete if and only if W_e is coinfinite. $\square$

The proof for IND is similar, but my proof for IST is a lot more complicated.

Index Set Results

Theorem

IST is Π^0_3*-complete within CSC.*

Proof sketch.

Fix $e \in \omega$. Let $W_e = \{x_0, x_1, x_2, \dots\}$. For each s, let ℓ_s be the least ℓ such that $[\ell, x_s] \subseteq W_{e,s}$. For all $i \in \omega$, let

$$V_{2i} = \{s \in \omega : \ell_s \leq i\}$$
$$V_{2i+1} = \{s \in \omega : \ell_s < \ell_i \text{ or } (\ell_s = \ell_i \text{ and } x_s \leq x_i)\}.$$

Verifying that this works requires proving two key facts:

- $\lim_s \ell_s = \infty$ if and only if W_e is coinfinite.
- If W_e is coinfinite, then for all i there is j such that $V_{2i} = V_{2j+1}$.

□

Outline

Computable Categoricity

Definition

Let X be a CSC space.

- The **weak computable dimension** of X is the number of computable copies of X up to computable homeomorphism.
- X is **weakly computably categorical** if X has weak computable dimension 1.

Theorem

$\mathbb{N}_{IND}$ *and* $\mathbb{N}_{DIS}$ *are weakly computably categorical.*

Proof.

Let X be a computable indiscrete CSC space. Any computable bijection $X \to \mathbb{N}_{IND}$ is a computable homeomorphism. The analogous fact holds for $\mathbb{N}_{DIS}$. $\square$

Effective Homeomorphisms

Definition (Dorais 2011)

Let $(X, \mathcal{U}, k)$ and $(Y, \mathcal{V}, \ell)$ be CSC spaces.

- A function $f : X \to Y$ is **effectively continuous** if f is computable and there is a computable function Φ such that for all x and i, if $f(x) \in V_i$, then $x \in U_{\Phi(x,i)} \subseteq f^{-1}(V_i)$.
- A function $f : X \to Y$ is an **effective homeomorphism** if f is a bijection and both f and f^{-1} are effectively continuous.

Definition

Let X be a CSC space.

- The **computable dimension** of X is the number of computable copies of X up to effective homeomorphism.
- X is **computably categorical** if X has computable dimension 1.

Why *Effective* Homeomorphisms?

1. Quantifiers:
   - For computable structures $\mathcal{A}$ and $\mathcal{B}$, the statement "$\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic" is Σ^0_3.
   - For computable CSC spaces X and Y, the statement "X and Y are computably homeomorphic" is Σ^0_4.
   - For computable CSC spaces X and Y, the statement "X and Y are effectively homeomorphic" is Σ^0_3.
2. Effective homeomorphisms preserve effective properties.

Discrete Topology

Definition

- For a CSC space $(X, \mathcal{U}, k)$, a **discreteness function** for X is a function $d : X \to \omega$ such that $U_{d(x)} = \{x\}$ for all $x \in X$.
- A CSC space is **effectively discrete** if it has a computable discreteness function.

Lemma

Let X and Y be effectively homeomorphic CSC spaces, and let d_X be a discreteness function for X. Then Y has a discreteness function d_Y computable from d_X. In particular, if X is effectively discrete, then so is Y.

Discrete Topology

Fact

There are computable discrete CSC spaces which are not effectively discrete (see e.g. Dorais 2011 and Benham et al. 2024).

Proposition

$\mathbb{N}_{DIS}$ is not computably categorical.

Theorem

For each $e \in \omega$, there is a computable discrete CSC space X_e such that X_e has a unique discreteness function d_e, and $d_e \equiv_T W_e$.

Corollary

$\mathbb{N}_{DIS}$ has computable dimension ω.

Initial Segment Topology

Fact

If X is homeomorphic to $\mathbb{N}_{IST}$, then the homeomorphism is unique.

Theorem

For each e such that W_e is noncomputable, there is a computable CSC space X_e such that X_e has the initial segment topology, and the Turing degree of the unique homeomorphism $X_e \to \mathbb{N}_{IST}$ is the same as that of W_e.

Corollary

$\mathbb{N}_{IST}$ is not weakly computably categorical.

Categoricity Results

In summary:

CSC Space	Weak Computable Dimension	Computable Dimension
$\mathbb{N}_{IND}$	1	1
$\mathbb{N}_{DIS}$	1	ω
$\mathbb{N}_{IST}$	ω	ω

Further Questions

Investigate computable categoricity for other CSC spaces:

- Cofinite topology on $\mathbb{N}$
- $\mathbb{Q}$ with the Euclidean topology

Question

Are there any other (non-trivial) CSC spaces with computable dimension 1?

Question

What should it mean for a CSC space to be computably categorical relative to a degree $\mathbf{d}$?

Bonus Result

Question

Is there a CSC space with no computable presentation?

Definition

Let α be a countable ordinal. Write α_{IST} for the CSC space $(\alpha, (\beta)_{\beta<\alpha}, k)$ where $k(x, i, j) = \min(i, j)$.

Proposition

If α is a noncomputable ordinal, then α_{IST} has no computable presentation.

Proof idea.

Suppose $(\omega, \mathcal{U}, k) \cong \alpha_{IST}$. For $a, b \in \omega$, say that $a <_\alpha b$ if and only if $\exists i(a \in U_i \wedge b \notin U_i)$. Then $(\omega, <_\alpha) \cong \alpha$ as linear orders.

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